

# NON-ABELIAN SYMMETRIES OF QUASITORIC MANIFOLDS

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ABSTRACT. A quasitoric manifold  $M$  is a  $2n$ -dimensional manifold which admits an action of an  $n$ -dimensional torus which has some nice properties. We determine the isomorphism type of a maximal compact connected Lie-subgroup  $G$  of  $\text{Homeo}(M)$  which contains the torus. Moreover, we show that this group is unique up to conjugation.

## 1. INTRODUCTION

A quasitoric manifold is a smooth connected orientable  $2n$ -dimensional manifold  $M$  with a smooth action of an  $n$ -dimensional torus  $T$  such that:

- The  $T$ -action on  $M$  is locally standard, i.e. the  $T$ -action is locally modelled on the standard  $T$ -action on  $\mathbb{C}^n$ .
- If the first property is satisfied, then  $M/T$  is naturally an  $n$ -dimensional manifold with corners. We require that the orbit space of the  $T$ -action on  $M$  is face-preserving homeomorphic to an  $n$ -dimensional simple polytope.

Quasitoric manifolds were introduced by Davis and Januszkiewicz [1] in 1991. A symplectic  $2n$ -dimensional manifold with an hamiltonian action of an  $n$ -dimensional torus is an example of a quasitoric manifold. We call such a manifold a symplectic toric manifold.

In this paper we construct a maximal compact connected Lie-subgroup of the homeomorphism group of  $M$  which contains the torus  $T$ . To be more precise, we have the following theorem.

**Theorem 1.1.** *Let  $M$  be a quasitoric manifold. Then there is a compact connected Lie-subgroup  $G$  of  $\text{Homeo}(M)$  which contains the torus  $T$  such that:*

- (1)  $G$  acts smoothly on  $M$  for some smooth structure on  $M$ .
- (2) If  $G' \subset \text{Homeo}(M)$  is another compact connected Lie-subgroup which contains the torus  $T$  and acts smoothly on  $M$  for some smooth structure on  $M$ , then  $G'$  is conjugated in  $\text{Homeo}(M)$  to a subgroup of  $G$ .
- (3) If the  $G$ - and  $G'$ -actions are smooth with respect to the same smooth structure on  $M$ , then  $G'$  is conjugated in  $\text{Diff}(M)$  to a subgroup of  $G$ .
- (4) If  $M$  is a symplectic toric manifold, then  $G$  is conjugated in  $\text{Homeo}(M)$  to a subgroup of the symplectomorphism group of  $M$ .

A smooth structure on  $M$  for which the  $G$ -action from the above theorem is smooth can be described as follows. By Theorem 5.6 of [4] the  $T$ -equivariant smooth structures on  $M$  correspond one-to-one to smooth structures on the orbit space  $M/T$ . The  $G$ -action is smooth for the  $T$ -equivariant smooth structure on  $M$  for which  $M/T$  is diffeomorphic to a simple polytope.

In this paper all actions of compact Lie-groups on manifolds  $M$  are smooth with respect to some smooth structure on  $M$ .

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This article is organized as follows. In section 2 we review some basic facts about quasitoric manifolds and introduce an automorphism group for the characteristic pair corresponding to a quasitoric manifold. In section 3 we construct the group  $G$  from Theorem 1.1. In section 4 we review the classification of quasitoric manifolds with  $G$ -action up to  $G$ -equivariant diffeomorphism given in [5]. Moreover, we give a classification of these manifolds up to  $G$ -equivariant homeomorphism. In section 5 we apply the results of the previous section and show that the group  $G$  has the properties described in Theorem 1.1.

## 2. CHARACTERISTIC PAIRS AND THEIR AUTOMORPHISM GROUPS

Let  $M$  be a  $2n$ -dimensional quasitoric manifold,  $P$  its orbit polytope and  $\pi : M \rightarrow P$  the orbit map. Denote by  $\mathfrak{F}(M)$  the set of facets of  $P$ . We write also  $\mathfrak{F}$  instead of  $\mathfrak{F}(M)$  if it is clear from the context which quasitoric manifold is meant. Then the preimage  $M_i = \pi^{-1}(F_i)$  of  $F_i \in \mathfrak{F}$  is a codimension two submanifold of  $M$  which is fixed by a one dimensional subtorus  $\lambda(F_i) = \lambda(M_i)$  of  $T$ . These  $M_i$  are called characteristic submanifolds of  $M$ . Since the facets of  $P$  correspond one-to-one to the characteristic submanifolds of  $M$ , we denote the set of characteristic manifolds also by  $\mathfrak{F}$ .

Let  $IT \subset LT$  be the integral lattice in the Lie algebra of  $T$ . The characteristic map  $\lambda : \mathfrak{F} \rightarrow \{\text{one-dimensional subtori of } T\}$  lifts to a map  $\bar{\lambda} : \mathfrak{F} \rightarrow IT \cong \mathbb{Z}^n$  such that, for a subset  $\sigma$  of  $\mathfrak{F}$  with  $\bigcap_{F_i \in \sigma} F_i \neq \emptyset$ ,  $\{\bar{\lambda}(F_i); F_i \in \sigma\}$  is part of a basis of  $IT$ . Note that each  $\bar{\lambda}(M_i)$  is unique up to sign. We call  $\bar{\lambda}$  a characteristic function for  $M$ .

Dual to  $P$  there is a simplicial complex  $K$  with vertex set  $\mathfrak{F}$ . A subset  $\sigma \subset \mathfrak{F}$  is a simplex of  $K$  if and only if  $\bigcap_{F_i \in \sigma} F_i \neq \emptyset$ .

Note that  $M$  is determined by the combinatorial type of  $P$  (or  $K$ ) and  $\bar{\lambda}$  up to equivariant homeomorphism [1, Proposition 1.8]. This construction motivates the following definition.

**Definition 2.1.** Let  $K$  be a simplicial complex of dimension  $n - 1$  with vertex set  $\mathfrak{F}$ . Moreover let  $T$  be an  $n$ -dimensional torus and  $\bar{\lambda} : \mathfrak{F} \rightarrow IT \cong \mathbb{Z}^n$  a map such that for all simplices  $\sigma$  of  $K$  the set  $\{\bar{\lambda}(F_i); F_i \in \sigma\}$  is part of a basis of  $IT$ . Then we call  $(K, \bar{\lambda})$  a characteristic pair.

An omniorientation of a quasitoric manifold  $M$  is a choice of orientations for  $M$  and all characteristic submanifolds of  $M$ . An omniorientation of  $M$  induces a complex structure on all normal bundles of the characteristic submanifolds. These complex structures may be used to make the map  $\bar{\lambda}$  unique by requiring that the  $S^1$ -action induced by  $\bar{\lambda}(M_i)$  on the normal bundle of  $M_i$  is given by complex multiplication.

The cohomology  $H^*(M; \mathbb{Z})$  was computed by Davis and Januszkiewicz [1, Theorem 4.14]. It is torsion-free and generated by the Poincaré-duals  $PD(M_i)$  of the characteristic manifolds. If we choose  $\bar{\lambda}$  as above, then these Poincaré-duals are subject to the following relations

$$(2.1) \quad 0 = \sum_{M_i \in \mathfrak{F}} \langle v, \bar{\lambda}(M_i) \rangle PD(M_i) \quad \text{for all } v \in IT^*$$

and, for  $\sigma \subset \mathfrak{F}$ ,

$$0 = \prod_{M_i \in \sigma} PD(M_i) \quad \Leftrightarrow \quad \sigma \text{ is not a simplex of } K.$$

If  $\sigma$  is a simplex of  $K$ , then, because  $\{\bar{\lambda}(F_i); F_i \in \sigma\}$  is part of a basis of  $IT$ , we may choose an isomorphism  $IT \rightarrow \mathbb{Z}^n$ , such that the  $\bar{\lambda}(F_i)$ 's are given by the

columns of a matrix of the form

$$\Lambda = \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & \Lambda' & \\ & & & 1 \end{pmatrix},$$

where the first  $\#\sigma$  columns of  $\Lambda$  correspond to the  $F_i \in \sigma$ . We call such a matrix a characteristic matrix of  $M$ . If  $v_1, \dots, v_n$  is the basis of  $IT^*$  dual to the standard basis of  $IT \cong \mathbb{Z}^n$ , then the coefficient  $\langle v_j, \bar{\lambda}(M_i) \rangle$  in equation (2.1) is the  $i$ -th entry of the  $j$ th row of  $\Lambda$ . Hence, one can read off all relations in (2.1) from the matrix  $\Lambda$ .

We are interested in the symmetries of  $M$ . Since quasitoric manifolds are determined by their characteristic pairs, we should also study automorphisms of characteristic pairs. Therefore we define:

**Definition 2.2.** Let  $(K, \bar{\lambda})$  be a characteristic pair. Then we define the automorphism group of  $(K, \bar{\lambda})$  to be

$$\text{aut}(K, \bar{\lambda}) = \{(f, g) \in \text{aut}(K) \times \text{aut}(T); Lg \circ \bar{\lambda} = \bar{\lambda} \circ f\}.$$

**Lemma 2.3.** Let  $M$  be a quasitoric manifold. Choose an omniorientation on  $M$  such that for two characteristic submanifolds  $M_1, M_2 \subset M$  we have

$$PD(M_1) = \pm PD(M_2) \Rightarrow PD(M_1) = PD(M_2).$$

For  $\alpha \in H^2(M)$  let

$$\mathfrak{F}_\alpha = \{M_i \in \mathfrak{F}; PD(M_i) = \alpha\}.$$

Then there is a unique homomorphism  $\phi : \prod_{\alpha \in H^2(M)} S(\mathfrak{F}_\alpha) \hookrightarrow \text{aut}(K, \bar{\lambda})$  such that  $\psi \circ \phi : \prod_{\alpha \in H^2(M)} S(\mathfrak{F}_\alpha) \hookrightarrow S(\mathfrak{F})$  is the standard inclusion. Here,  $\psi : \text{aut}(K, \bar{\lambda}) \rightarrow S(\mathfrak{F})$  is the natural projection.

*Proof.* Let  $\sigma \in \prod_{\alpha \in H^2(M; \mathbb{Z})} S(\mathfrak{F}_\alpha) \subset S(\mathfrak{F})$ . If  $(f, g)$  is an element of  $\text{aut}(K, \bar{\lambda})$  such that  $\psi((f, g)) = \sigma$ , then we must have  $f = \sigma$  and  $Lg(\bar{\lambda}(F_i)) = \bar{\lambda}(\sigma(F_i))$  for all  $F_i \in \mathfrak{F}$ . Since  $IT$  is generated by the  $\bar{\lambda}(F_i)$ ,  $F_i \in \mathfrak{F}$ ,  $g$  is uniquely determined by  $\sigma$ . Therefore a homomorphism  $\phi : \prod_{\alpha \in H^2(M)} S(\mathfrak{F}_\alpha) \hookrightarrow \text{aut}(K, \bar{\lambda})$  with the properties described in the lemma is unique, if it exists.

Now we show that  $\phi$  exists. Let  $\sigma \in \prod_{\alpha \in H^2(M; \mathbb{Z})} S(\mathfrak{F}_\alpha) \subset S(\mathfrak{F})$ . Then we have, for  $I \subset \mathfrak{F}$ ,

$$\bigcap_{F_i \in I} F_i = \emptyset \Leftrightarrow \prod_{M_i \in I} PD(M_i) = \prod_{M_i \in I} PD(\sigma(M_i)) = 0 \Leftrightarrow \bigcap_{F_i \in I} \sigma(F_i) = \emptyset.$$

Therefore  $\sigma$  is an automorphism of  $K$ .

Now let  $F_1, \dots, F_n \in \mathfrak{F}$  such that  $\bigcap_{i=1}^n F_i \neq \emptyset$ . Then  $\bar{\lambda}(F_1), \dots, \bar{\lambda}(F_n)$  is a basis of  $IT$ . Moreover, since  $\sigma$  is an automorphism of  $K$ , the same holds for  $\bar{\lambda}(\sigma(F_1)), \dots, \bar{\lambda}(\sigma(F_n))$ .

Therefore we may define an automorphism  $g_\sigma$  of  $T$  by  $Lg_\sigma(\bar{\lambda}(F_i)) = \bar{\lambda}(\sigma(F_i))$  for  $i = 1, \dots, n$ . Let  $v_1, \dots, v_n$  be the basis of  $IT^*$  dual to  $\bar{\lambda}(F_1), \dots, \bar{\lambda}(F_n)$  and  $v'_1, \dots, v'_n$  the basis of  $IT^*$  dual to  $\bar{\lambda}(\sigma(F_1)), \dots, \bar{\lambda}(\sigma(F_n))$ . Then we have, for  $i = 1, \dots, n$ ,

$$\begin{aligned} \sum_{M_j \in \mathfrak{F} - \{M_1, \dots, M_n\}} \langle v_i, \bar{\lambda}(M_j) \rangle PD(M_j) &= -PD(M_i) = -PD(\sigma(M_i)) \\ &= \sum_{M_j \in \mathfrak{F} - \{\sigma(M_1), \dots, \sigma(M_n)\}} \langle v'_i, \bar{\lambda}(M_j) \rangle PD(M_j) \\ &= \sum_{M_j \in \mathfrak{F} - \{M_1, \dots, M_n\}} \langle v'_i, \bar{\lambda}(\sigma(M_j)) \rangle PD(M_j) \end{aligned}$$

Since  $\{PD(M_i); M_i \in \mathfrak{F} - \{M_1, \dots, M_n\}\}$  is a basis of  $H^2(M)$ , it follows that  $Lg_\sigma(\bar{\lambda}(M_i)) = \bar{\lambda}(\sigma(M_i))$  for all  $M_i \in \mathfrak{F}$ .

Therefore  $(\sigma, g_\sigma) \in \text{aut}(K, \bar{\lambda})$  and we define  $\phi(\sigma) = (\sigma, g_\sigma)$ .  $\square$

**Lemma 2.4.** *Let  $M$  be a quasitoric manifold as in Lemma 2.3. Then for  $x \in M^T$  and  $\alpha \in H^2(M)$  we have*

$$\#\mathfrak{F}_\alpha - 1 \leq \#\{M_i \in \mathfrak{F}_\alpha; x \in M_i\} \leq \#\mathfrak{F}_\alpha$$

*Proof.* It follows from the relations (2.1) that the Poincaré duals of the  $M_i \in \mathfrak{F}$ ,  $x \notin M_i$ , form a basis of  $H^2(M)$ .  $\square$

### 3. CONSTRUCTING GROUP ACTIONS

In this section we construct an action of a compact connected Lie-group on a quasitoric manifold which extends the torus action.

Before we do that we explain how an action of a compact connected Lie-group on a quasitoric manifold  $M$  induces a homomorphism of the Weyl-group  $W(G) \rightarrow \text{aut}(K, \bar{\lambda})$ . Here and in the following we choose an omniorientation of  $M$  as in Lemma 2.3.

Assume that there is a compact connected Lie-group  $G$  such that a finite quotient of  $G$  acts on  $M$  by an extension of the  $T$ -action. Then  $M$  is called a quasitoric manifold with  $G$ -action.

By Lemma 2.8 of [5],  $G$  has a covering group of the form  $\prod_{i=1}^k SU(l_i + 1) \times T^{l_0}$ . Moreover, if  $g \in N_G T$  and  $M_i \in \mathfrak{F}$ , then  $gM_i$  is a characteristic submanifold of  $M$ .

Therefore we get an action of  $W(G)$  on  $\mathfrak{F}$ . Since  $G$  is connected, this action identifies  $W(G)$  with a subgroup of  $\prod_{\alpha \in H^2(M)} S(\mathfrak{F}_\alpha) \subset S(\mathfrak{F})$ . There are disjoint subsets  $\mathfrak{F}_1, \dots, \mathfrak{F}_k$  of  $\mathfrak{F}$  such that  $W(SU(l_i + 1))$  is identified with  $S(\mathfrak{F}_i) \subset S(\mathfrak{F})$  for  $i = 1, \dots, k$  [5, Section 2]. We call  $\mathfrak{F}_i$  the set of characteristic submanifolds of  $M$  which are permuted by  $W(SU(l_i + 1))$ .

Moreover,  $W(G)$  acts on  $T$  by conjugation. The actions of  $W(G)$  on  $T$  and  $\mathfrak{F}$  induce an action of  $W(G)$  on the characteristic pair  $(K, \bar{\lambda})$ . The homomorphism  $W(G) \rightarrow \text{aut}(K, \bar{\lambda})$  corresponding to this action is the restriction of the homomorphism  $\phi$  constructed in Lemma 2.3 to  $W(G)$ .

Now we state the main theorem of this section.

**Theorem 3.1.** *Let  $M$  be a quasitoric manifold as in Lemma 2.3. Then there is a smooth structure and a smooth action of a compact connected Lie-group  $G$  on  $M$  which extends the torus action such that  $W(G) = \prod_{\alpha \in H^2(M)} S(\mathfrak{F}_\alpha)$ .*

*Proof.* We prove this theorem by induction on the dimension of  $M$ . If  $\dim M = 0$  or  $\#\mathfrak{F}_\alpha \leq 1$  for all  $\alpha \in H^2(M)$ , then there is nothing to prove. Therefore assume that there is an  $\alpha \in H^2(M)$  such that  $\#\mathfrak{F}_\alpha \geq 2$ .

By Lemmas 2.3 and 2.4, there are two cases:

- (1)  $\bigcap_{M_i \in \mathfrak{F}_\alpha} M_i = \emptyset$  and, for all  $M_{i_0} \in \mathfrak{F}_\alpha$ ,  $\bigcap_{M_i \in \mathfrak{F}_\alpha - \{M_{i_0}\}} M_i \neq \emptyset$ ,
- (2)  $\bigcap_{M_i \in \mathfrak{F}_\alpha} M_i \neq \emptyset$ .

We consider at first the case (1). In this case we have with  $N = \bigcap_{M_i \in \mathfrak{F}_\alpha - \{M_{i_0}\}} M_i$  that

- (1)  $M/T = \Delta^{\#\mathfrak{F}_\alpha - 1} \times N/T$  and

(2)

$$\Lambda(M) = \begin{pmatrix} 1 & & -1 & 0 & \dots & 0 \\ & \ddots & \vdots & \vdots & \ddots & \vdots \\ & & 1 & -1 & 0 & \dots & 0 \\ 0 & \dots & 0 & \lambda_1 & & & \\ \vdots & \ddots & \vdots & \vdots & & \Lambda(N) & \\ 0 & \dots & 0 & \lambda_k & & & \end{pmatrix},$$

where  $\Lambda(M)$  and  $\Lambda(N)$  are the characteristic matrices of  $M$  and  $N$ , respectively. Here the first columns correspond to the facets in  $\mathfrak{F}_\alpha$ . They are of the form  $F \times N/T$ , where  $F$  runs through the facets of  $\Delta^{\#\mathfrak{F}_\alpha-1}$ .

We show at first (1). If this is shown then (2) follows immediately from (2.1). Let  $M_i \in \mathfrak{F}(M) - \mathfrak{F}_\alpha$ , then by Lemmas 2.3 and 2.4 we must have  $N \cap M_i \neq \emptyset$ . Therefore we have a bijection  $\mathfrak{F}(M) - \mathfrak{F}_\alpha \rightarrow \mathfrak{F}(N)$ . Because, by the same lemmas, the intersection of  $M_{i_1}, \dots, M_{i_k} \in \mathfrak{F}(M) - \mathfrak{F}_\alpha$  is empty if and only if  $N \cap \bigcap_{j=1}^k M_{i_j} = \emptyset$ , (1) follows.

Therefore, by Proposition 1.8 of [1],  $M$  is equivariantly homeomorphic to

$$S^{2\#\mathfrak{F}_\alpha-1} \times_{S^1} N.$$

Here the action of  $S^1$  on  $N$  is induced by the homomorphism  $\phi : S^1 \rightarrow T'$  to the torus, which acts on  $N$ , defined by  $\mu = (\lambda_1, \dots, \lambda_k)^t = \sum_{M_i \in \mathfrak{F}_\alpha} \bar{\lambda}(M_i)$ . Moreover,  $S^1$  acts on  $S^{2\#\mathfrak{F}_\alpha-1} \subset \mathbb{C}^{2\#\mathfrak{F}_\alpha}$  by multiplication.

By the induction hypothesis there is a compact connected Lie-group  $G'$  which acts on  $N$  by an extension of the torus action, such that  $G'$  realizes the action of  $\prod_{\beta \in H^2(N)} S(\mathfrak{F}_\beta)$  on the simplicial complex dual to  $N/T$ .

Then the action of  $SU(\#\mathfrak{F}_\alpha) \times G'$  on  $S^{2\#\mathfrak{F}_\alpha-1} \times N$  induces an action of  $G = SU(\#\mathfrak{F}_\alpha) \times_{Z_{G'}(\phi(S^1))} G'$  on  $M$  such that the action of  $G/H$  extends the torus action. Here  $Z_{G'}(\phi(S^1))$  is the centralizer of  $\phi(S^1)$  in  $G'$  and  $H$  is the ineffective kernel of the  $G$ -action on  $M$ .

From Lemma 2.5 of [5] we know that  $W(SU(\#\mathfrak{F}_\alpha)) = S(\mathfrak{F}_\alpha)$ . Therefore we have to show that

$$W(Z_{G'}(\phi(S^1))) = \prod_{\beta \in H^2(M) - \{\alpha\}} S(\mathfrak{F}_\beta(M)) \subset \prod_{\beta \in H^2(N)} S(\mathfrak{F}_\beta(N)) = W(G').$$

It follows from the remarks at the beginning of this section that  $W(Z_{G'}(\phi(S^1)))$  is a subgroup of  $\prod_{\beta \in H^2(M) - \{\alpha\}} S(\mathfrak{F}_\beta(M))$ . Therefore let  $w \in \prod_{\beta \in H^2(M) - \{\alpha\}} S(\mathfrak{F}_\beta(M))$ . Since  $\prod_{\beta \in H^2(M) - \{\alpha\}} S(\mathfrak{F}_\beta(M))$  is generated by transpositions, we may assume that  $w$  is a transposition.

Let  $M_1, \dots, M_{m'} \in \mathfrak{F}(M) - \mathfrak{F}_\alpha$  such that  $\bigcap_{i=1}^{m'} M_i \cap N$  is a single point. Then  $\bar{\lambda}(M_1), \dots, \bar{\lambda}(M_{m'})$  form a basis of  $IT'$ . Let  $v_1, \dots, v_{m'}$  be the dual basis of  $IT'^*$ . Let  $i \in \{1, \dots, m'\}$ . At first assume that  $w(M_i) = M_{i'}$  with  $i' \in \{1, \dots, m'\}$ . Then we have

$$\begin{aligned} \langle v_i, \mu \rangle \alpha + \sum \langle v_i, \bar{\lambda}(M_j) \rangle PD(M_j) &= -PD(M_i) = -PD(M_{i'}) \\ &= \langle v_{i'}, \mu \rangle \alpha + \sum \langle v_{i'}, \bar{\lambda}(M_j) \rangle PD(M_j). \end{aligned}$$

Here the sums are taken over those characteristic submanifolds of  $M$  which do not belong to  $\mathfrak{F}_\alpha \cup \{M_1, \dots, M_{m'}\}$ . Their Poincaré duals together with  $\alpha$  form a basis of  $H^2(M)$ . Therefore we have

$$\langle v_i, \mu \rangle = \langle v_{i'}, \mu \rangle = \langle w^* v_i, \mu \rangle = \langle v_i, w_* \mu \rangle.$$

Now assume that  $w(M_i) \neq M_1, \dots, M_{m'}$ . Then, because  $PD(M_i) = PD(w(M_i))$  we must have  $\langle v_i, \bar{\lambda}(M_j) \rangle = 0$  for all  $M_j \in \mathfrak{F}(M) - \{M_i, w(M_i)\}$ . This implies  $\langle v_i, \mu \rangle = 0$ . Moreover, we have

$$\begin{aligned} \langle v_i, w_*\mu \rangle &= \sum_{j=1}^{m'} \langle v_i, w_*\bar{\lambda}(M_j) \rangle \langle v_j, \mu \rangle \\ &= \sum_{j \in \{1, \dots, m'\} - \{i\}} \langle v_i, \bar{\lambda}(M_j) \rangle \langle v_j, \mu \rangle = 0. \end{aligned}$$

Therefore we have  $\mu = w_*\mu$ . This implies  $w \in W(Z_{G'}(\phi(S^1)))$ . Hence, the claim follows in this case.

Now assume that  $\bigcap_{M_i \in \mathfrak{F}_\alpha} M_i$  is non-empty. Then let  $\tilde{M}$  be the blow-up of  $M$  along  $\bigcap_{M_i \in \mathfrak{F}_\alpha} M_i$  (see Section 4 of [5] for details). If we write the characteristic matrix of  $\tilde{M}$  in the form

$$(3.1) \quad \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \Lambda' \\ & & & 1 \end{pmatrix}$$

such that the first  $\#\mathfrak{F}_\alpha$  columns correspond to the  $F_i \in \mathfrak{F}_\alpha$ , then the characteristic matrix of  $\tilde{M}$  is given by

$$\begin{pmatrix} 1 & & & & & -1 \\ & \ddots & & & & \vdots \\ & & 1 & & & -1 \\ & & & 1 & & \Lambda' \\ & & & & 1 & 0 \\ & & & & & \ddots \\ & & & & & 1 & 0 \end{pmatrix},$$

where the proper transforms of the characteristic submanifolds of  $M$  are ordered as in (3.1), the last column corresponds to the exceptional submanifold and the first  $\#\mathfrak{F}_\alpha$  entries in this column are equal to  $-1$ .

Hence two characteristic submanifolds of  $M$  have the same Poincaré-duals if and only if their proper transforms have the same Poincaré-duals. Moreover, the Poincaré-dual of the exceptional submanifold is distinct from the Poincaré-duals of the other characteristic submanifolds of  $\tilde{M}$ .

By the first case there is a  $G$ -action on  $\tilde{M}$  which extends the torus action. Since the exceptional submanifold is fixed by  $\phi(S^1)$ , we can  $G$ -equivariantly blow down  $\tilde{M}$  along the exceptional manifold to get a  $G$ -action on  $M$  (see [5, Section 4] for details).  $\square$

**Corollary 3.2.** *The group  $G$  constructed in Theorem 3.1 has a covering group of the form  $\prod_{\alpha \in H^2(M; \mathbb{Z})} SU(\#\mathfrak{F}_\alpha) \times T^{l_0}$ .*

*Proof.* This follows from the results of Section 2 of [5] and the description of the  $W(G)$ -action on  $\mathfrak{F}$  given in Theorem 3.1.  $\square$

In [4] we proved that the equivariant smooth structures on a quasitoric manifold correspond one-to-one to the smooth structures on its orbit space. We will show that the  $G$ -action constructed in Theorem 3.1 is smooth with respect to the smooth structure on  $M$  which corresponds to the natural smooth structure on the simple polytope  $P$ . This will follow from the proof of Theorem 3.1, Corollary 5.3 of [4] and the following lemma.

**Lemma 3.3.** *In the situation of Theorem 5.16 of [5] we have:  $N/T$  is diffeomorphic to a simple polytope if and only if  $M/T$  is diffeomorphic to a simple polytope.*

*Proof.* If  $M/T$  is diffeomorphic to a simple polytope, then  $N/T$  is diffeomorphic to a simple polytope because  $N/T$  is a face of  $M/T$ .

So we only have to prove the other implication. If  $N/T$  is diffeomorphic to a simple polytope, then all face-preserving homeomorphisms constructed in the proof of the cited theorem may be replaced by diffeomorphisms. So if we follow the proof of this theorem we end-up with a diffeomorphism

$$g : F_1 \times \Delta^{l_1} \rightarrow F_1 \times \Delta^{l_1},$$

which we want to extend to a diffeomorphism  $F_1 \times \Delta^{l_1+1} \rightarrow F_1 \times \Delta^{l_1+1}$ . That this is possible follows from Theorem 5.1 of [4] because every facet of  $F_1 \times \Delta^{l_1}$  of the form  $F_1 \times F$ , where  $F$  is a facet of  $\Delta^{l_1}$ , is mapped by  $g$  to a facet of the same form.  $\square$

**Corollary 3.4.** *The action of the group  $G$  constructed in Theorem 3.1 is smooth with respect to the smooth structure on  $M$  corresponding to the natural smooth structure on  $P$ .*

*Proof.* We use the same induction and notations as in the proof of Theorem 3.1. If the  $G'$ -action on  $N$  is smooth with respect to the smooth structure on  $N$  for which  $N/T$  is diffeomorphic to a simple polytope, then by Lemma 3.3 the  $G$ -action on  $M$  is smooth with respect to the smooth structure for which  $M/T$  is diffeomorphic to a simple polytope. Since two simple polytopes are combinatorially equivalent if and only if they are diffeomorphic [4, Corollary 5.3], it follows that the  $G$ -action on  $M$  is smooth with respect to the smooth structure for which  $M/T$  is diffeomorphic to  $P$ .  $\square$

#### 4. CLASSIFICATION

Let  $G$  be a compact Lie-group with maximal torus  $T$ . A quasitoric manifold with  $G$ -action is a smooth  $G$ -manifold  $M$  such that  $M$  together with the action of the maximal torus of  $G/H$  is a quasitoric manifold. Here  $H$  is a finite subgroup of  $G$  which acts trivially on  $M$ . In [5] we classified quasitoric manifolds with  $G$ -action. As a first step towards this classification we showed that  $G$  has a covering group of the form  $\prod_{i=1}^k SU(l_i + 1) \times T^{l_0}$ .

The classification was given in terms of admissible triples  $(\psi, N, (A_1, \dots, A_k))$ , where

- $\psi$  is an homomorphism  $\prod_{i=1}^k S(U(l_i) \times U(1)) \rightarrow T^{l_0}$ .
- $N$  is a  $2n$ -dimensional quasitoric manifold.
- The  $A_i$  are characteristic submanifolds of  $N$  or empty. If  $A_i$  is non-empty then  $\text{im } \psi|_{S(U(l_i) \times U(1))}$  acts trivially on  $A_i$  and  $\ker \psi|_{S(U(l_i) \times U(1))} = SU(l_i)$ .

Two such triples  $(\psi, N, (A_1, \dots, A_k))$  and  $(\psi', N', (A'_1, \dots, A'_k))$  are called equivalent or diffeomorphic if

- $\psi|_{S(U(l_i) \times U(1))} = \psi'|_{S(U(l_i) \times U(1))}$  if  $l_i > 1$ .
- $\psi|_{S(U(l_i) \times U(1))} = \psi'^{\pm 1}|_{S(U(l_i) \times U(1))}$  if  $l_i = 1$ .
- There is an  $T^{l_0}$ -equivariant diffeomorphism  $f : N \rightarrow N'$  such that  $f(A_i) = A'_i$  for all  $i$ .

The main theorem of [5] may be formulated in the following way:

**Theorem 4.1** ([5, Theorem 8.6]). *Let  $G = \prod_{i=1}^k SU(l_i + 1) \times T^{l_0}$ . Then the  $G$ -equivariant diffeomorphism classes of quasitoric manifolds with  $G$ -action are in one-to-one correspondence with the diffeomorphism classes of admissible triples.*

We call two admissible triples  $(\psi, N, (A_1, \dots, A_k))$  and  $(\psi', N', (A'_1, \dots, A'_k))$  homeomorphic if

- $\psi|_{S(U(l_i) \times U(1))} = \psi'|_{S(U(l_i) \times U(1))}$  if  $l_i > 1$ .
- $\psi|_{S(U(l_i) \times U(1))} = \psi'^{\pm 1}|_{S(U(l_i) \times U(1))}$  if  $l_i = 1$ .
- There is a  $T^{l_0}$ -equivariant homeomorphism  $f : N \rightarrow N'$  such that  $f(A_i) = A'_i$  for all  $i$ .

With this notation we have the following classification of quasitoric manifolds with  $G$ -action up to  $G$ -equivariant homeomorphism.

**Theorem 4.2.** *Let  $G = \prod_{i=1}^k SU(l_i + 1) \times T^{l_0}$ . Then the  $G$ -equivariant homeomorphism classes of quasitoric manifolds with  $G$ -action are in one-to-one correspondence with the homeomorphism classes of admissible triples.*

*Proof.* At first note that the homeomorphism type of the admissible triple corresponding to a quasitoric manifold with  $G$ -action depends only on the  $G$ -equivariant homeomorphism type of  $M$  because  $N$  and the  $A_i$  may be identified with intersections of characteristic submanifolds of  $M$  and  $\psi$  depends only on the isotropy groups of points in these intersections (see Sections 5 and 8 of [5] for details).

Because by the proof of Corollary 8.8 of [5]  $M/G = N/T^{l_0}$ , it follows from Lemma 4.3 below that the homeomorphism type of the admissible triple determines the  $G$ -equivariant homeomorphism type of  $M$ .  $\square$

**Lemma 4.3.** *Let  $M$  be a quasitoric manifold with  $G$ -action,  $G = \prod_{i=1}^k SU(l_i + 1) \times T^{l_0}$ . Then  $M$  is equivariantly homeomorphic to  $M/G \times G/\sim$ , where  $(x, g) \sim (x', g')$  if and only if  $x = x'$  and  $gg'^{-1} \in H_x$ . Here the groups  $H_x$  depend only on  $x$  and the admissible triple corresponding to  $M$ .*

*Proof.* We prove this lemma by induction on the number of simple factors of  $G$ . If  $G$  is a torus then this lemma is due to Davis and Januszkiewicz [1, Proposition 1.8].

Therefore we may assume that there is at least one simple factor. There are two cases:

- (1)  $M^{SU(l_1+1)} = \emptyset$
- (2)  $M^{SU(l_1+1)} \neq \emptyset$ .

In the first case we have by Corollary 5.6 of [5]

$$M = SU(l_1 + 1) \times_{S(U(l_1) \times U(1))} M',$$

where  $M'$  is a quasitoric manifold with  $G'$ -action,  $G' = \prod_{i=2}^k SU(l_i + 1) \times T^{l_0}$ . The action of  $S(U(l_1) \times U(1))$  on  $N$  is induced by the homomorphism

$$\phi = (\psi|_{S(U(l_1) \times U(1))})^{-1} : S(U(l_1) \times U(1)) \rightarrow T^{l_0},$$

where  $\psi$  is the homomorphism from the admissible triple of  $M$ . Moreover, by the proof of Corollary 8.8 of [5], we have  $M/G = M'/G'$ . Therefore by the induction hypothesis we have

$$M = SU(l_1 + 1) \times_{S(U(l_1) \times U(1))} (M/G \times G'/\sim').$$

For a subgroup  $H'_x$  of  $G'$  we have

$$SU(l_1 + 1) \times_{S(U(l_1) \times U(1))} (G'/H'_x) = (SU(l_1 + 1) \times G')/(\phi \text{Id}_{G'})^{-1}(H'_x).$$

Therefore the statement follows in this case with  $H_x = (\phi \text{Id}_{G'})^{-1}(H'_x)$ .

If  $M^{SU(l_1+1)}$  is non-empty. Then  $M$  is the blow-down of some quasitoric manifold  $\tilde{M}$  with  $G$ -action but without  $SU(l_1 + 1)$  fixed points. Let  $F : \tilde{M} \rightarrow M$  be the projection. Then we have  $M = \tilde{M}/\sim''$ , where  $y \sim'' y'$  if and only if there is a



$g \in SU(l_1 + 1)$  such that  $gy = y'$  in the case  $y, y' \in F^{-1}(M^{SU(l_1+1)})$  or  $y = y'$  otherwise.

Since under the identification of  $M/G$  with  $N/T^{l_0}$  given in the proof of Corollary 8.8 of [5]  $M^{SU(l_1+1)}/G$  is identified with  $A_1/T^{l_0}$ . for  $x \in M^{SU(l_1+1)}/G$ , we have  $\text{im } \phi = \text{im } \psi \subset H'_x$ .

Therefore we have  $S(U(l_1) \times U(1)) \times H'_x = (\phi \text{Id}_{G'})^{-1}(H'_x)$  if  $x \in M^{SU(l_1+1)}/G$ . Hence the statement follows with

$$H_x = \begin{cases} SU(l_1 + 1) \times H'_x & \text{if } x \in M^{SU(l_1+1)}/G \\ (\phi \text{Id}_{G'})^{-1}(H'_x) & \text{if } x \notin M^{SU(l_1+1)}/G. \end{cases}$$

□

**Theorem 4.4.** *Let  $G = \prod_{i=1}^k SU(l_i + 1) \times T^{l_0}$  and  $M, M'$  be two quasitoric manifolds with  $G$ -action. Let  $T$  be a maximal torus of  $G$ . Then  $M$  and  $M'$  are  $G$ -equivariantly homeomorphic (diffeomorphic), if and only if they are  $T$ -equivariantly homeomorphic (diffeomorphic).*

*Proof.* Without loss of generality we may assume that  $T$  is the standard maximal torus of  $G$ . Let  $(\psi, N, (A_1, \dots, A_k))$  be the admissible triple corresponding to  $M$ . We show that  $(\psi, N, (A_1, \dots, A_k))$  is determined up to homeomorphism (diffeomorphism) by the homeomorphism (diffeomorphism) type of the  $T$ -action on  $M$ .

At first, by Lemmas 2.7 and 2.10 of [5], the characteristic submanifolds which are permuted by the Weyl-group  $W(SU(l_i + 1))$  are exactly those characteristic manifolds  $M_j$  for which  $\pi_i \circ \lambda(M_j)$  is non-trivial. Here  $\pi_i : G \rightarrow SU(l_i + 1)$  is the projection. Denote by  $\mathfrak{F}_i$  the set of the characteristic submanifolds which are permuted by  $W(SU(l_i + 1))$ .

If  $l_i > 1$ , there is exactly one  $M_{j_i} \in \mathfrak{F}_i$  such that  $\lambda(M_{j_i})$  is fixed by  $W(S(U(l_i) \times U(1)))$ . Because for each  $M_{j_0} \in \mathfrak{F}_i$  we have  $\dim \langle \pi_i \circ \lambda(M_j); M_j \in \mathfrak{F}_i - \{M_{j_0}\} \rangle = l_i$ ,  $M_{j_i}$  is the only characteristic submanifold such that  $\pi_i \circ \lambda(M_{j_i})$  is contained in the center of  $S(U(l_i) \times U(1))$ .

If  $l_i = 1$ , then  $\mathfrak{F}_i$  has exactly two elements and any choose of a  $M_{j_i} \in \mathfrak{F}_i$  leads to the same equivalence class of admissible triples (see [5, Section 5] for details).

Then we have

$$N = \bigcap_{i=1}^k \bigcap_{M_j \in \mathfrak{F}_i - \{M_{j_i}\}} M_j \quad \text{and} \quad A_i = N \cap M_{j_i}.$$

By its construction in the proof of Lemma 5.3 of [5], the homomorphism  $\psi$  depends only on the  $\prod_{i=1}^k S(U(l_i) \times U(1)) \times T^{l_0}$ -representation  $T_x M$  with  $x \in N^{T^{l_0}}$ . Since  $T$  is a maximal torus of  $\prod_{i=1}^k S(U(l_i) \times U(1)) \times T^{l_0}$ , this representation depends only on the  $T$ -equivariant homeomorphism type of  $M$ .

Now the statement follows from Theorem 4.2. □

## 5. UNIQUENESS

In this section we prove that the group constructed in section 3 is a maximal compact connected Lie-subgroup of the homeomorphism group of  $M$  which contains the torus and that it is unique up to conjugation.

**Lemma 5.1.** *Let  $M$  be a quasitoric manifold with  $G$ -action,  $G = \prod_{i=1}^k SU(l_i + 1) \times T^{l_0}$  and  $T$  a maximal torus of  $G$ , such that  $T^{l_0}$  acts effectively on  $M$ . Denote by  $\mathfrak{F}_i, i = 1, \dots, k$ , the set of characteristic submanifolds of  $M$  which are permuted by  $W(SU(l_i + 1))$ . Moreover, let  $\mathfrak{F}_0 = \mathfrak{F} - \bigcup_{i=1}^k \mathfrak{F}_i$ . Then we have:*

(1) The subgroup of  $G$  which acts trivially on  $M$  is given by

$$H = \{(g, \psi(g)) \in G; g \in Z(\prod_{i=1}^k SU(l_i + 1))\}.$$

(2) Let  $M_1$  be a characteristic submanifold of  $M$  which belongs to  $\mathfrak{F}_i$  and  $x \in M_1$  a generic point. Then  $T_x$  is connected and

$$H \cap T_x = \{(g, \psi(g)) \in G; g \in Z(SU(l_i + 1))\}$$

if  $i > 0$ . If  $i = 0$  then  $T_x$  is connected and  $H \cap T_x = 1$ .

*Proof.* At first we prove (1). We prove this statement by induction on  $k$ . If  $k = 0$ , then there is nothing to prove. Therefore assume that  $k > 0$  and that the statement is proved for all quasitoric manifolds with  $G' = \prod_{i=2}^k SU(l_i + 1) \times T^{l_0}$ -action. With the notation from the proof of Lemma 4.3 the subgroup of  $G$  which acts trivially on  $M$  is given by

$$\begin{aligned} H &= \bigcap_{x \in M/G, g \in G} gH_xg^{-1} = \bigcap_{x \in M/G, g \in G} g(\phi \text{Id}_{G'})^{-1}(H'_x)g^{-1} \\ &= \bigcap_{g \in SU(l_1+1)} g(\phi \text{Id}_{G'})^{-1} \left( \bigcap_{x \in M/G, g' \in G'} g'H'_xg'^{-1} \right) g^{-1} \\ &= \bigcap_{g \in SU(l_1+1)} g \langle \{(h, \psi(h)); h \in S(U(l_1) \times U(1))\}, \bigcap_{x \in M/G, g' \in G'} g'H'_xg'^{-1} \rangle g^{-1} \\ &= \langle \{(h, \psi(h)); h \in Z(SU(l_1 + 1))\}, H' \rangle \\ &= \{(g, \psi(g)) \in G; g \in Z(\prod_{i=1}^k SU(l_i + 1))\}. \end{aligned}$$

Here  $H'$  denote the subgroup of  $G'$  which acts trivially on  $N$ .

Now we prove the second statement. At first assume  $i > 0$ . After blowing up  $M$  along the fixed points of  $SU(l_i + 1)$ , we may assume that

$$M = SU(l_i + 1) \times_{S(U(l_i) \times U(1))} N.$$

Then there is an  $SU(l_i + 1)$ -equivariant projection  $p : M \rightarrow \mathbb{C}P^{l_i}$ . The characteristic submanifold  $M_1$  of  $M$  is given by a preimage of a characteristic submanifold  $\mathbb{C}P_1^{l_i}$  of  $\mathbb{C}P^{l_i}$ . Now we have

$$T_x = \{(t, \psi(t)) \in T; t \in T_{p(x)}^{l_i}\},$$

where  $T^{l_i}$  denotes  $T \cap SU(l_i + 1)$ . Since  $T_{p(x)}^{l_i}$  contains the center of  $SU(l_i + 1)$  the statement follows in this case.

If  $i = 0$ , then it follows from Lemma 2.10 of [5] that  $T_x$  is contained in  $T^{l_0}$ . Therefore the statement follows in this case.  $\square$

**Lemma 5.2.** *Let  $M$  be a quasitoric manifold and  $T$  the torus which acts on  $M$  by  $\phi : T \rightarrow \text{Homeo}(M)$ . Let  $G_j$ ,  $j = 1, 2$ , be compact connected Lie-groups and  $\iota_j : T \rightarrow G_j$ ,  $j = 1, 2$ , embeddings of  $T$  as maximal tori of  $G_j$ . Assume that there are effective actions  $\phi_j : G_j \rightarrow \text{Homeo}(M)$ ,  $j = 1, 2$ , such that  $\phi = \phi_j \circ \iota_j$  for  $j = 1, 2$ . Moreover, assume that the natural actions of  $W(G_j)$ ,  $j = 1, 2$ , on  $\mathfrak{F}$  induce identifications of  $W(G_1)$  and  $W(G_2)$  with a given subgroup  $H$  of  $S(\mathfrak{F})$ . Then  $\phi_1(G_1)$  and  $\phi_2(G_2)$  are conjugated in  $\text{Homeo}(M)$ .*

*Proof.* Since the Weyl-groups of  $G_1$  and  $G_2$  are isomorphic. There is a group of the form  $\tilde{G} = \prod_{i=1}^k SU(l_i + 1) \times T^{l_0}$  and coverings  $\varphi_j : \tilde{G} \rightarrow G_j$ ,  $j = 1, 2$ .

Because all maximal tori in  $\tilde{G}$  are conjugated, we may assume that there is a maximal torus  $\tilde{T}$  of  $\tilde{G}$  such that  $\tilde{T} = \varphi_j^{-1}(\iota_j(T))$ ,  $j = 1, 2$ . Let  $\psi$  be the automorphism of  $L\tilde{T}$  given by  $(L\varphi_1)^{-1} \circ L\iota_1 \circ (L\iota_2)^{-1} \circ L\varphi_2$ .

Choose an omniorientation for  $M$  such that, for all characteristic submanifolds  $M_i, M_j$  of  $M$ , we have:

$$PD(M_i) = \pm PD(M_j) \Rightarrow PD(M_i) = PD(M_j).$$

This omniorientation is preserved by the actions of  $G_1$  and  $G_2$ . Denote by  $\bar{\lambda}$  the characteristic function for the  $T$ -action on  $M$ . Moreover, denote by  $\bar{\lambda}_j(M_i)$ ,  $j = 1, 2$ ,  $M_i \in \mathfrak{F}$ , a primitive vector in  $I\tilde{T}$  which generates the isotropy group of a generic point in  $M_i$  with respect to the  $\tilde{T}$ -action  $\phi_j \circ \varphi_j$ . We choose this primitive vector in such a way that it is compatible with the omniorientation chosen above.

Then by Lemma 5.1 we have, for all  $M_k \in \mathfrak{F}_i$ ,  $i > 0$ , and  $j = 1, 2$ ,

$$L\iota_j^{-1} \circ L\varphi_j(\bar{\lambda}_j(M_k)) = (l_i + 1)\bar{\lambda}(M_k).$$

For  $M_k \in \mathfrak{F}_0$  we have

$$L\iota_j^{-1} \circ L\varphi_j(\bar{\lambda}_j(M_k)) = \bar{\lambda}(M_k).$$

This implies that  $\psi(\bar{\lambda}_2(M_k)) = \bar{\lambda}_1(M_k)$ . Therefore for  $w \in W(\tilde{G})$  we have by Lemma 2.10 of [5]

$$(5.1) \quad \psi(\bar{\lambda}_2(wM_k)) = \bar{\lambda}_1(wM_k) = w\bar{\lambda}_1(M_k)w^{-1} = w\psi(\bar{\lambda}_2(M_k))w^{-1}$$

$$(5.2) \quad \bar{\lambda}_2(wM_k) = w\bar{\lambda}_2(M_k)w^{-1}.$$

It follows that  $\psi$  is an automorphism of the  $W(\tilde{G})$ -representation  $L\tilde{T}$ . Because each irreducible non-trivial summand of  $L\tilde{T}$  appears only once in a decomposition of  $L\tilde{T}$  in irreducible representations, it follows from Schur's Lemma that the restriction of  $\psi$  to the Lie-algebra of the maximal torus  $\tilde{T}_i$  of a simple factor  $SU(l_i + 1)$  of  $\tilde{G}$  is multiplication with a constant  $a_i \in \mathbb{R}$ .

Therefore we have

$$\iota_1^{-1} \circ \varphi_1(\tilde{T}_i) = \iota_2^{-1} \circ \varphi_2(\tilde{T}_i).$$

Denote this subtorus of  $T$  by  $T_i$ . By Lemma 5.1, we have that

$$IT_i/L\iota_1^{-1} \circ L\varphi_1(I\tilde{T}_i) \cong \ker \iota_1^{-1} \circ \varphi_1 \cap \tilde{T}_i \cong \ker \iota_2^{-1} \circ \varphi_2 \cap \tilde{T}_i \cong IT_i/L\iota_2^{-1} \circ L\varphi_2(I\tilde{T}_i).$$

Note that  $I_{ij} = \langle \bar{\lambda}_j(M_k); M_k \in \mathfrak{F} \rangle \cap I\tilde{T}_i$  is a lattice of maximal rank in  $I\tilde{T}_i$ . Then we have

$$\begin{aligned} |I\tilde{T}_i/I_{i1}| &= \frac{|IT_i/L\iota_1^{-1} \circ L\varphi_1(I_{i1})|}{|IT_i/L\iota_1^{-1} \circ L\varphi_1(I\tilde{T}_i)|} \\ &= \frac{|IT_i/L\iota_2^{-1} \circ L\varphi_2(I_{i2})|}{|IT_i/L\iota_2^{-1} \circ L\varphi_2(I\tilde{T}_i)|} \\ &= |I\tilde{T}_i/I_{i2}| = \frac{1}{|a_i|} |I\tilde{T}_i/I_{i1}|, \end{aligned}$$

because  $\psi(I_{i2}) = I_{i1}$ . Therefore we must have  $a_i = \pm 1$ . Therefore there is an automorphism  $\Psi$  of  $\tilde{G}$  with  $L\Psi = \psi$ . Now the statement follows from Theorem 4.4 applied to the  $\tilde{G}$ -actions  $\phi_1 \circ \varphi_1 \circ \Psi$  and  $\phi_2 \circ \varphi_2$ .  $\square$

*Remark 5.3.* If, in the situation of Lemma 5.2, both  $G_j$ -actions are smooth with respect to the same smooth structure, then it follows from Theorem 4.4 that  $\phi_1(G_1)$  and  $\phi_2(G_2)$  are conjugated in  $\text{Diff}(M)$ .

**Theorem 5.4.** *Let  $M$  be a quasitoric manifold and  $G \subset \text{Homeo}(M)$ , the group constructed in section 3. If  $G'$  is an other compact connected Lie-group which acts by an extension of the torus action on  $M$ . Then  $G'$  is conjugated in  $\text{Homeo}(M)$  to a subgroup of  $G$ . If the  $G$  and  $G'$ -actions are smooth for the same smooth structure on  $M$ , then  $G'$  is conjugated in  $\text{Diff}(M)$  to a subgroup of  $G$ .*

*Proof.* Let  $\mathfrak{F} = \mathfrak{F}_1 \amalg \dots \amalg \mathfrak{F}_k$  be a partition in  $W(G')$ -orbits. Then we have  $W(G') = \prod_{i=1}^k S(\mathfrak{F}_i)$ . Moreover since the  $G'$ -action on  $H^*(M)$  is trivial it follows that the sets  $\mathfrak{F}_\alpha$ ,  $\alpha \in H^2(M)$ , are  $W(G')$ -invariant.

This gives as an homomorphism  $W(G') \rightarrow W(G) = \prod_{\alpha \in H^2(M)} S(\mathfrak{F}_\alpha)$ . There is a subgroup of maximal rank of  $G$  whose Weyl-group is given by the image of this homomorphism. Therefore the statement follows from Lemma 5.2.  $\square$

Now we have proven all parts of Theorem 1.1 besides the statement about the symplectic toric manifolds. To prove this part we first recall the construction of a maximal compact Lie-subgroup of the symplectomorphism group of a symplectic toric manifold due to Masuda [2]. An alternative construction of this group was given by McDuff and Tolman [3].

He showed that there is a root system  $R(M)$  such that the root system of every compact connected Lie-subgroup of the symplectomorphism group which contains the torus is a subroot system of  $R(M)$ . Moreover, he constructed a compact Lie-subgroup  $G'$  of the symplectomorphism group which contains the torus and has a root system isomorphic to  $R(M)$ .

The proof of the first part of Masuda's results is also valid for any compact connected Lie-subgroup of the homeomorphism group of  $M$  which contains the torus and preserves the omniorientation induced by the symplectic form on  $M$ . Therefore  $G$  and  $G'$  are conjugated, if the  $G$ -action preserves this omniorientation.

Hence, it is sufficient to prove that if  $M_1$  and  $M_2$  are characteristic submanifolds of  $M$  with  $PD(M_1) = \pm PD(M_2)$ , then we have  $PD(M_1) = PD(M_2)$ . We consider two cases  $M_1 \cap M_2 = \emptyset$  and  $M_1 \cap M_2 \neq \emptyset$ . We should note here that if  $M$  is a symplectic toric manifold then  $\bar{\lambda}(F_i)$  is the outward normal vector of the facet  $F_i$  of  $P$ .

Let  $e_1, \dots, e_n$  be the standard basis of  $\mathbb{R}^n$ . In the first case we may assume that  $\{0\} = F_1 \cap \bigcap_{i=3}^{n+1} F_i \subset \mathbb{R}^n$  and  $\bar{\lambda}(F_1) = e_1$ ,  $\bar{\lambda}(F_i) = e_{i-1}$  for  $i = 3, \dots, n+1$ . It follows from  $PD(M_1) = \pm PD(M_2)$  that  $\bar{\lambda}(F_2) = \pm e_1 + \sum_{i=2}^n \mu_{i2} e_i$  and  $\bar{\lambda}(F_j) = \sum_{i=2}^n \mu_{ij} e_i$  for  $j > n+1$  with  $\mu_{ij} \in \mathbb{Z}$ . Therefore  $P \cap \langle e_1 \rangle$  is an interval with boundary  $\langle e_1 \rangle \cap (F_1 \cup F_2)$ . Hence we must have  $\bar{\lambda}(F_2) = -e_1 + \sum_{i=2}^n \mu_{i2} e_i$ . This implies  $PD(M_1) = PD(M_2)$ .

Now consider the case  $M_1 \cap M_2 \neq \emptyset$ .

Without loss of generality we may assume that  $\{0\} = \bigcap_{i=1}^n F_i \subset \mathbb{R}^n$  and  $\bar{\lambda}(F_i) = e_i$  for  $i = 1, \dots, n$ .

Assume that  $PD(M_1) = -PD(M_2)$ . Then for all  $F_j \in \mathfrak{F}$ ,  $j > n$ , there are  $\mu_{0j}, \mu_{3j}, \dots, \mu_{nj} \in \mathbb{Z}$  such that

$$\bar{\lambda}(F_j) = \mu_{0j}(e_1 - e_2) + \sum_{i=3}^n \mu_{ij} e_i.$$

Because the  $\bar{\lambda}(F_j)$  are the outward normal vectors of the facets of  $P$  it follows that  $P \cap \langle e_1, e_2 \rangle$  is non-compact. But this is impossible because  $P$  is a convex polytope. Therefore we must have  $PD(M_1) = PD(M_2)$ .

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